Study of Local Level Spacing in Random Matrix Theory

Dissertation submitted to
Visvesvaraya National Institute of Technology, Nagpur
in partial fulfilment of the requirements for the award of degree of

Master of Science
In
Physics

By

Himanshu Chaudhary
(MS23PHY006)

under the supervision of

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Department of Physics
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DEPARTMENT OF PHYSICS

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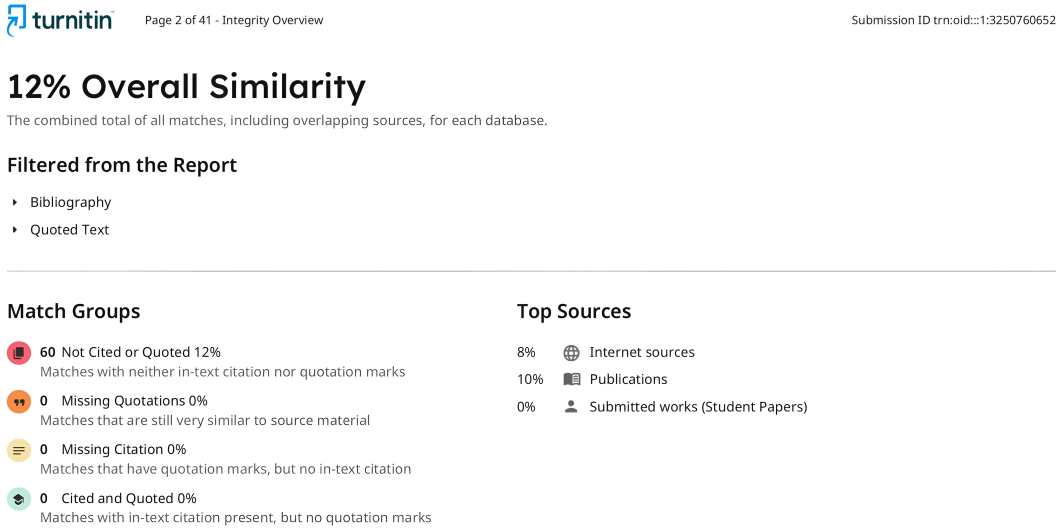
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Abstract

In this thesis, we explore the study of local level spacings and their ratios, which are integral to the spectral analysis of complex systems using random matrix theory(RMT). We examine the behavior of various random matrix ensembles and provide distributions for higher-order ratios of their spacings. The primary goal of this research is to analyze the behavior of random eigenvalues of Circular ensembles, both with and without reference points, and their impact on the average values of spacing ratios. We generate distributions for consecutive spacing ratios as well as higher-order spacing ratios, highlighting changes in average values that influence the observed spectral data. Our findings indicate that the average values of consecutive spacing ratios with reference points do not correspond to the analytical formula provided by E. Bogomolny for cases without reference points.

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Chapter 1

Introduction to Random Matrix Theory

Random Matrix Theory, as the name implies, deals with the concept of randomness in matrices. Let us now delve into what it entails and explore the methods through which we study this fascinating framework. It is one of those rare tools that connects applied mathematics with physics and biology. In our universe, there exist numerous systems that are both chaotic and complex, whose behavior cannot be predicted by any conventional method. In such systems, even a slight change in the initial conditions can lead to dramatically different outcomes in their trajectories. To analyze and understand the behavior of these systems, we require a mathematical framework, which we refer to as Random Matrix Theory.

1.1 History

Random Matrix Theory has its roots in the middle of the 20th century, first in the field of nuclear physics. The concept was first introduced by Eugene Wigner, an American physicist, in 1950. He was examining the energy levels of heavy atomic nuclei, which were too complex to analyze using classical methods. He proposed that the spacings between these energy levels could be modeled using the eigenvalues of large random matrices. This idea made the birth of Random Matrix Theory [5].

Wigner's groundbreaking work laid the foundation for Random Matrix Theory, and it was later taken up by other researchers. For instance, Freeman Dyson developed the mathematical framework for random matrix ensembles, introducing concepts like the Gaussian Orthogonal Ensemble(GOE), Gaussian Unitary Ensemble(GUE) , and Gaussian Symplec-

tic Ensemble(GSE). These ensembles became crucial to the study of Random Matrix Theory [5].

In beginning, Random Matrix Theory was confined to nuclear physics, but now, its applications spread to various fields, including quantum chaos, number theory, and even modern areas like machine learning and finance. The universality of Random Matrix Theory predictions has made it a powerful tool for understanding complex systems across all sciences.

1.2 Need of Random Matrix Theory

Systems like triple pendulum and big nucleus do behave very complex outputs of their trajectory and that is non-predictable while the future behavior has nothing to take with the previous trajectories likewise in general, when a small change produces large deviation in the future trajectories of a system, then such a type of system is called chaotic system. To study them there was no method, so then the renowned scientist of that time gave a concept of corelation of two independent data space, consider two matrices that contain the data set of marks of three different subjects now to check if there is a corelation between the two data sets, and we can write the data set as in matrix form now the corelation matrix will be the product of the matrix with its transpose with that concept embarks the sight of for developing a mathematical model and there was the need of the Random Matrix theory [16].

So now the mathematical model must contain the random values which lie in different probability distributions and after mathematically formulate them in probability distribution functions. We can have a study of the eigenvalues of such complex systems by the spacing distribution and how they behave in the symmetries of time and space.

1.3 Requirements in all over science

The million dolor Math question has to do with each other, well nothing almost, atomic nuclei are physical systems in which neutrons and protons interact with each other meanwhile billiards are a common example of systems study in chaos theory a field which deals with dynamical systems that are sensitive to small changes and the million dolor Maths question

is the Reimann zeta hypothesis. They all are surprisingly well described by random matrices because they all share a common trait of being random [17].

Random matrix theory connects to explain variety of fields of science that are useful in daily life. In physics the topics it cover to explain are Chaos theory, string theory, gauge theory, black holes, quantum circuits, thermalization, statistical mechanics. While in mathematics it is used in number theory, topology, complex analysis, Riemann hypothesis, while it has also uses in Biology that are protein folding, mankov chains, neural networks etc.

1.4 Application of Random Matrix Theory

Finance

As we see there are many takeaway and random orientation in the market of stocks. So lets say we have a corelation matrix between the stocks it tells you how much each stock is linked to other stocks, we can not get actually peculiar while only staring on to the matrices we got so we go mathematically, we find out the eigenvalues and eigenvectors of those co-related matrices, where eigenvectors shows main pattern of movement in the market.

These findings of the correlation helps to breakdown the market as mentioned by the Marčenko-Pastur in their paper [8], acoording to Marčenko-Pastur law, when the true correlation matrix is the identity that means no correlation between the variables, the density of eigenvalues $\rho(\lambda)$ of the empirical correlation matrix is given, in the limit of large matrices, by the Marčenko-Pastur distribution: that is given by :

$$\rho(\lambda) = \frac{Q}{2\pi}\frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{\lambda} \quad \text{for} \quad \lambda \in [\lambda_-, \lambda_+], \tag{1.1}$$

where,

$$\lambda_\pm = (1 \pm \sqrt{1/Q})^2 \quad \text{and} \quad Q = \frac{T}{N}.$$

(the ratio of the number of observations to the number of variables). Where, $\rho(\lambda)$ is the probability distribution of the eigenvalues,while λ_-, λ_+ are the minimum and maximum eigenvalues expected for pure noise. Q=$\frac{T}{N}$ is ratio of number of time points T to number of assets N. So if your eigenvalues lies between these two maximum and minimum values then

this is noise otherwise eigenvalues it is real information. Marčenko-Pastur law predicted the exact shape of the spectrum of eigenvalues when you only have noise [8].

Biology

In the field of diffusion magnetic resonance imaging(DMRI), that is a powerful technique that maps the diffusion of water molecules in biological tissues providing insights into micro structural organization of the brain, however this data of images contain noise that are mainly of three type, thermal noise, physiological noise and motion artifacts, how do we get to denoise them and get accurate data, so in this process of denoising the use of Random matrix theory is there, first we take the image data and form it in the matrix form where the entries are voxel's signal now we use the method to identify the noise same as we done by the formulation of Marčenko-Pastur that uses correlation matrices to know the noise and real signals [9].

Although the study primarily emphasizes a subset of commonly used diffusion metrics, the proposed methodology is broadly applicable to all diffusion parameters, whether empirically derived or estimated through biophysical modeling, since noise is a fundamental property of the raw signal and propagates into all derived measures. The results demonstrate that denoising not only improves signal fidelity but also exerts both local and global influences on tractography, sometimes producing changes that are indistinguishable from true anatomical structures [9].

Chapter 2

Ensembles and their Density Functions

2.1 Gaussian Orthogonal Ensemble

The Gaussian Orthogonal Ensemble (GOE) is a combination of real symmetric random matrices where the distribution of matrix elements is formulated by a Gaussian probability density function or normal distributions. It is mainly used in random matrix theory and in statistical physics related to quantum chaos [5].

Let H be an $N \times N$ real symmetric matrix:

$$H = H^{\top},$$

where $H^{\top}$ is the transpose matrix.

2.2 Probability Density Function for gaussian orthogonal ensemble

Each matrix H in the ensemble is distributed according to the following density:

$$P(H) \propto \exp\left(-\frac{N}{4}\mathrm{tr}(H^2)\right). \tag{2.1}$$

Properties associated with the matrix:

- Matrices are real symmetric i.e $H = H^{\top}$,

- Invariant under orthogonal transformations: $H \to O^\top HO$,
- Used in modeling spectra of complex systems (e.g., big nucleus).

2.3 Gaussian Unitary Ensemble

The Gaussian Unitary Ensemble are of complex Hermitian random matrices. Each matrix H in the ensemble satisfies [6]:

$$H = H^\dagger,$$

where $H^\dagger$ is the Hermitian conjugate (complex conjugate transpose) of H.

Probability Density Function for gaussian unitary ensemble

Each matrix H is distributed according to the following probability density [6]:

$$P(H) \propto \exp\left(-\frac{N}{2}\mathrm{tr}(H^2)\right). \tag{2.2}$$

More precisely, for an $N \times N$ matrix H with entries:

- $H_{ii} \sim N(0, \sigma^2)$ (real diagonal elements),
- $H_{ij}, H_{ij} \sim N(0, \sigma^2/2)$ for $i < j$ (complex off-diagonal),

Now we have,

$$P(H) = C_N \exp\left(-\frac{1}{\sigma^2}\mathrm{tr}(H^2)\right), \tag{2.3}$$

where C_N is a normalization constant.

2.4 Gaussian symplectic ensemble

The gaussian symplectic ensemble (GSE) is described by the gaussian measure with density

$$P(H) \propto \exp\left(-N\mathrm{tr}(H^2)\right). \tag{2.4}$$

In the space of $N \times N$ Hermitian quaternionic matrices, e.g, symmetric square matrices composed of quaternions. Their distribution is invariant under conjugation of the symmetric

group, and it models Hamiltonion with time-reversal symmetry [5]. Quaternions are 4 dimesnional complex numbers for example

$$q = ai + bj + ck + d, \tag{2.5}$$

where i,j,k are the negative unity in root and a,b,c,d belongs to real group of numbers. These numbers are useful in 3D rotations, quantum physics, and more abstractly in symplectic geometry.

2.5 Eigenvalue Distribution

Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of H. Then their joint probability density is:

$$P(\lambda_1, \ldots, \lambda_N) = C'_N \prod_{i<j} |\lambda_i - \lambda_j|^2 \exp\left(-\frac{1}{\sigma^2} \sum_{i=1}^{N} \lambda_i^2\right). \tag{2.6}$$

Compared to GOE, the power of the Vandermonde determinant is squared here, which corresponds to a higher degree of eigenvalue repulsion due to unitary symmetry [6].

2.6 Wigner's Semicircle Law

In the large N limit, the eigenvalue density converges to the Wigner semicircle law:

$$\rho(\lambda) = \begin{cases} \dfrac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 N - \lambda^2} & \text{if } |\lambda| \le 2\sigma\sqrt{N}. \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Wigner introduced this surmise to study energy levels of heavy atomic nuclei. Experimental spectra showed level repulsion, a signature character captured by the Wigner distribution. Surprisingly, the nontrivial zeros of the Riemann zeta function have spacings that exactly matches the Wigner distributions [6].

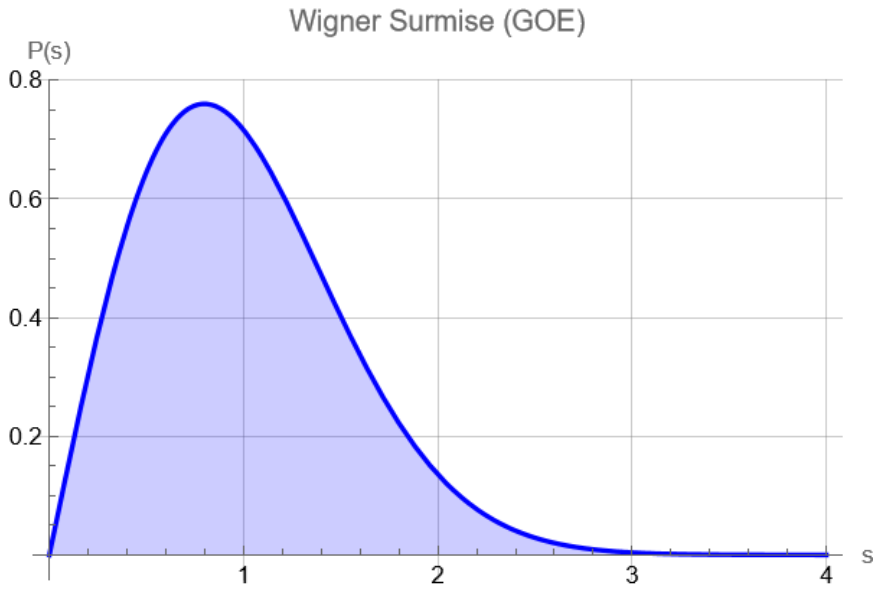

Figure 2.1: Surmise

properties associated with the matrix :

- Matrices are complex Hermitian: $H = H^{\dagger}$,
- Invariant under unitary transformations: $H \rightarrow U^{\dagger}HU$ for unitary U,
- Stronger eigenvalue repulsion compared to Gaussian orthogonal ensemble,
- Appears in quantum systems where time-reversal symmetry is broken.

2.7 Circular Ensembles the three fold way

Let consider a system which is characterized by a unitary matrix S, whose elements give the transition probabilities between the various states but not by its Hamiltonian H. The matrix S is unitary and so its eigenvalues can therefore take the form $e^{i\theta_j}$ where the angles θ_j are real and lie between the interval of 0 and 2π. The matrix S is a function of the Hamiltonian H of the system. The functional dependence need not be specified. All that is required is for small ranges of variation, the θ_j be linear functions of the eigenvalues of H [5].

$$P(\theta_1, \ldots, \theta_N) \propto \prod_{1 \leq j < k \leq N} \left| e^{i\theta_j} - e^{i\theta_k} \right|^{\beta}, \tag{2.8}$$

is the probability distribution of eigen angles in the circular ensemble. The circular ensembles and Gaussian ensembles have some sort of similarities as well as differences, as these are two important classes of random matrix theory, which is used to study statistical properties of eigenvalues of different complex systems. The biggest similarity between the two is that they both aim to model complex quantum systems and share universal statistical behaviors especially when the matrix size is very large, while the main difference lies in the matrix they involve, as gaussian ensembles do have matrices hermitian and symmetric they do give real eigenvalues and distributed on real line, while in the circular ensemble they consist of unitary matrix and their eigen angles lie on unit circle so while Gaussian ensembles are used in contexts involving energy levels, circular ensembles are more common in systems with periodicity and can be systems which are rotationally invariant.

Chapter 3

Spacing and Ratios of Spacing

3.1 Spacing

In the study of Random Matrix Theory, one of the most fundamental statistical quantities is the nearest-neighbor spacing, the difference between consecutive eigenvalues in a spectrum. This simple measurement captures deep insights into the structure and dynamics of complex quantum and statistical systems, from higher atomic nuclei to number theory. For instance, in the Riemann zeta function, the spacing between the nontrivial zeroes of the Riemann zeta function is studied through this method, and surprisingly, the distribution follows a distribution similar to random matrices [1]. Given ordered eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, the nearest neighbor spacing s_n is defined as:

$$s_n = \lambda_{n+1} - \lambda_n. \tag{3.1}$$

The distribution of these spacings tells us about the system's behavior. Spacing is an important parameter in random matrix theory because the distributions of these spacings reveal the underlying dynamics two main factors it tell us about the system:

- **Chaotic systems** In these types of systems the spacing between the eigenvalues can not be zero which means it shows level repulsion, where the eigenvalues avoid clustering too closely. This is seen in a distribution that vanishes as $s \to 0$.
- **Integrable systems**, In these often have uncorrelated spectra. Their spacing distribution follow an exponential law, allowing for small spacings to occur more frequently.

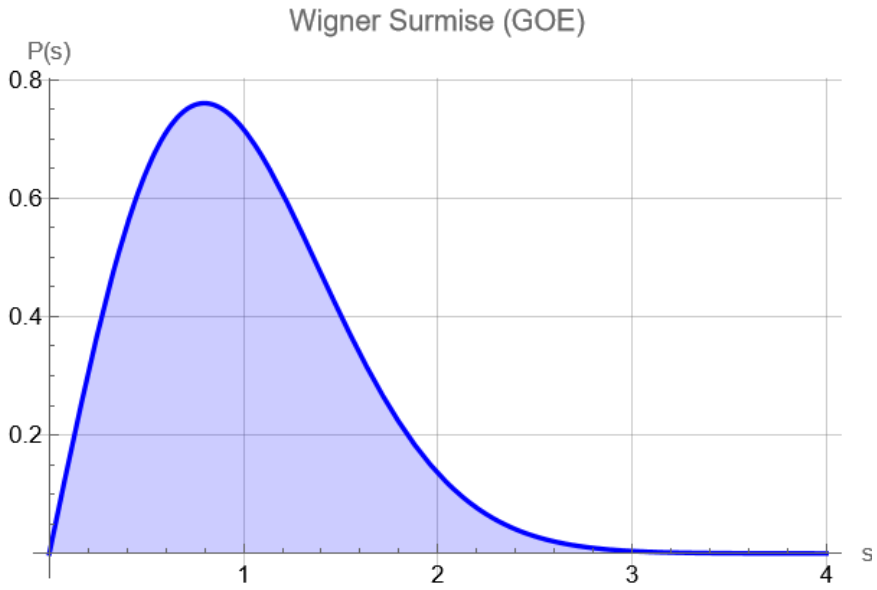

Figure 3.1: Chaotic spacing distribution

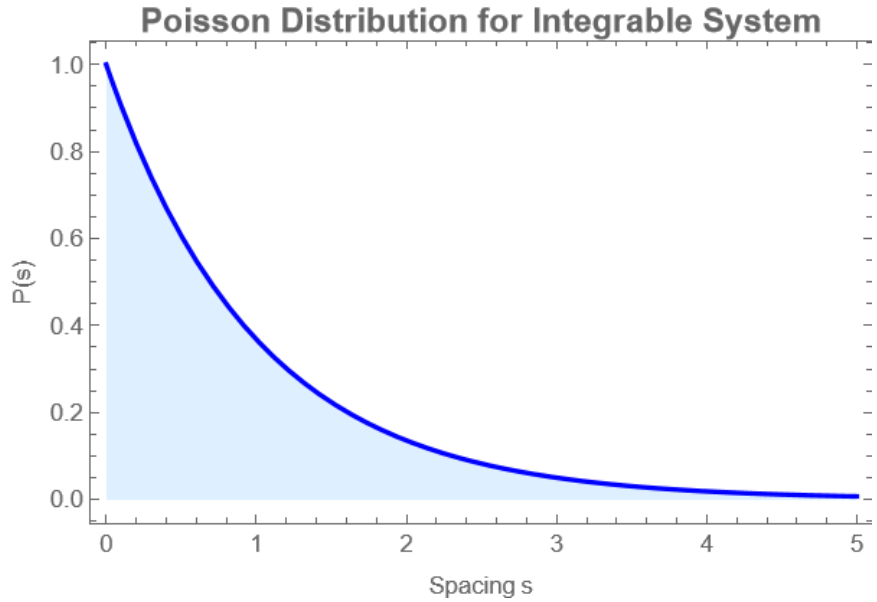

Figure 3.2: Integrable spacing distribution

In the study of dynamical systems, a clear distinction exists between integrable and chaotic behavior. Systems that are integrable tend to evolve in a predictable and orderly fashion, largely because they have a full set of conserved quantities that govern their motion. These systems follow patterns that can be solved through exact mathematical methods. Chaotic systems, however, behave in a much more unpredictable way. Even tiny changes in their starting conditions can lead to vastly different outcomes over time. Although their rules are deterministic, their future states are practically impossible to forecast after a certain point. This contrast plays a vital role in physics, where the structure of energy levels can reveal whether a system follows regular motion or descends into chaos.

3.2 Related Conjectures

Berry–Tabor Conjecture: If the classical dynamics of any system is fully integrable then the spacing between neighboring quantum energy levels tends to follow a Poisson distribution as one approaches the semiclassical regime [18]. This conjecture was given by Berry and Tabor in 1977 which is valid for integrable systems. $P(s)$ of finding a spacing s between adjacent levels follows:

$$P(s) = \exp(-s). \tag{3.2}$$

Bohigas–Giannoni–Schmit (BGS) Conjecture: Quantum systems with classically chaotic behavior exhibit spectral fluctuations that align closely with those anticipated by random matrix theory [19]. $P(s)$ of finding a spacing s between adjacent levels follows:

$$P(s) = \frac{\pi s}{2} \exp\left(-\frac{\pi s^2}{4}\right). \tag{3.3}$$

3.3 The Wigner Surmise

One of the earliest and most famous results in Random matrix theory is the Wigner surmise, it gives an approximation to the probability distribution $P(s)$ of spacings in the Gaussian Orthogonal Ensemble. The Wigner surmise gives accurate results for the nearest-neighbor level spacing distribution in the Gaussian Orthogonal Ensemble (GOE) of random matrix theory, especially for 2×2 matrices, and it also give a good approximation for larger matrices [6].

The formula is:

$$P(s) = \frac{\pi s}{2} \exp\left(-\frac{\pi s^2}{4}\right). \tag{3.4}$$

This expression takes the level repulsion behavior which means $P(s) \to 0$ as $s \to 0$, showing that closely spacedlevels are unlikely in chaotic systems. Different random matrix ensembles (GOE, GUE, GSE) have different spacing distributions, but they all share the feature of level repulsion. In contrast, for integrable systems, the spacing distribution is Poissonian:

$$P(s) = \exp(-s). \tag{3.5}$$

This lack of repulsion indicates uncorrelated levels, a hallmark of regular, predictable dynamics. A major condition in analyzing spacing distributions is that the raw spectra of a system often has varying density, the eigenvalues are not uniformly spaced on average. To make meaningful comparisons across different systems or ensembles, we have to unfold the spectrum. This process rescales the eigenvalues so that the mean local level density becomes uniform(normalized to one). Unfolding is a delicate, sometimes complex procedure, especially for systems with no clear analytic form for the level density. This makes spacing-based analysis sensitive to how unfolding is performed, potentially introducing errors. Even with the challenges, nearest-neighbor spacing remains a foundational tool in Random matrix theory. It gives immediate, intuitive insight into the nature of a spectrum. Also, it has inspired new tools, such as spacing ratios, which avoid unfolding altogether, but the idea of eigenvalue spacing remains central to understanding spectral statistics [20].

3.4 Ratios of Spacing

Traditional spectral studies include only the spacing to distinguish the system's spectra while there was a problem in it, during comparison in systems the unfolding is required, it involves estimating the local spectral density that can be noisy due to system behavior or can be more complex to evaluate so to remove this confusion of state, new form of evaluation was introduced by Oganesyan & Huse in 2007 [14]. The main idea behind this was to get rid of the need for unfolding, they introduced ratios of consecutive level spacings, such as:

$$r_n = \frac{s_n}{s_{n-1}} = \frac{\lambda_{n+1} - \lambda_n}{\lambda_n - \lambda_{n-1}}. \tag{3.6}$$

These ratios are dimensionless and do not depend on the local density of state [3]. It consist of following advantages:

- No need of unfolding the spectrum,
- Can be used directly on raw data,
- Much easier to compute and compare across complex and different systems.

The analysis of spectral statistics in quantum systems has increasingly favored the use of ratios of adjacent level spacings, as they provide a practical alternative to traditional methods. Unlike conventional spacing distributions, which typically demand the application of spectral unfolding, a process that can be ambiguous or unreliable in finite systems spacing ratios offers a more stable approach. First formalized in the work of Atas et al. (2013), these ratios (e.g., $r_n = \frac{s_{n+1}}{s_n}$) are dimensionless and capture key universal features of spectral correlations. For instance, different statistical ensembles yield distinct average values: approximately 0.386 for Poisson processes, 0.536 for the Gaussian Orthogonal Ensemble, and 0.602 for the Gaussian Unitary Ensemble. These ratios have proven especially useful in areas such as quantum chaos and many body localization, where robustness and minimal processing are essential. Their independence from local spectral density makes them a valuable tool for both analytical studies and computational simulations [20].

Chapter 4

Spacing With Fix Point

4.1 Dyson Circular Triad and angle spacing

In Figure 4.1, we can see that the eigenvalues of random unitary matrices can be spread on the unit circle. As the eigenvalues of the unitary matrix are always unity, the radius is always equal to 1, while we can spread the eigen angles from 0 to 2π, so that they look like the figure shown below [1].

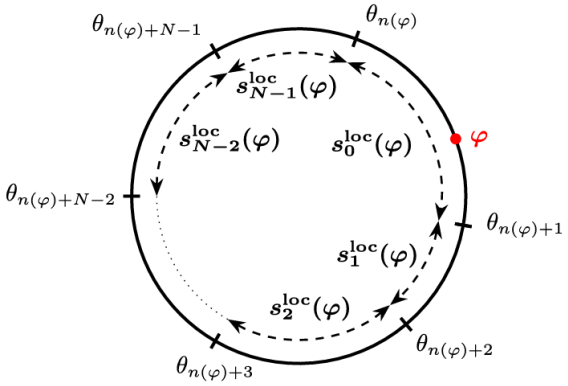

Figure 4.1: Eigen angles on circle

The concept of spacing with the fix point give us different results than the normal local level spacing results, but why do they behave like that, to understand that first we dive in to knowing what is the reference fix point situation. In here as in the figure we have distributed the eigen angles along the circle of unitary matrices, to be more specified these we extract

the matrix from circular orthogonal ensembles after that we calculate there eigenvalues then as we know that the eigenvalues of the unitary matrices can be sort and converted in to the eigen angles with range 0 to 2π, now in the figure a representation of such a system is shown where $\theta_{n(\varphi)}$, $\theta_{n(\varphi)+1}$, $\theta_{n(\varphi)+2}$ ... are the eigen angles that are distributed along the circumference of the unit circle.

Now the reference point come into the view where we choose a point between the whole range of angles, any arbitrary point we choose as the reference or say fix point on the circle now, the problem is if we take the average spacing between consecutive eigen angles of N such systems and on the other hand if we find the average spacing of the spacing where the reference point will lie, then the average value do not coincide in both. This phenomenon can be well explained by the inspection paradox. In the figure the reference point is shown by the φ as we can say that spacing that contains the point φ will have larger average value than that of the average consecutive spacing [1].

4.2 The Inspection Paradox

The inspection paradox is a counterintuitive phenomenon in probability theory, particularly within the study of renewal processes. It describes the situation in which, upon observing a process at a random point in time, the observed interval (such as the time between events) is more likely to be longer than the average interval. The strength of students is 20 only now if we find the average number of students then it is the total number of classes by total number of classes while if a person asks any random student that what the average strength of your class is, it would be most probably close to 100 now because as the number of students in class 9th and 10th are higher so the probability of asking a student at random is more in that range that phenomenon which changes the average value due to uneven distribution is called inspection paradox. So, here in our context of spacing distribution the probability of lying any point say reference we choose would be more in larger spacing so the average spacing is different from the intuitive spacing of total angle divided by the number of angles.

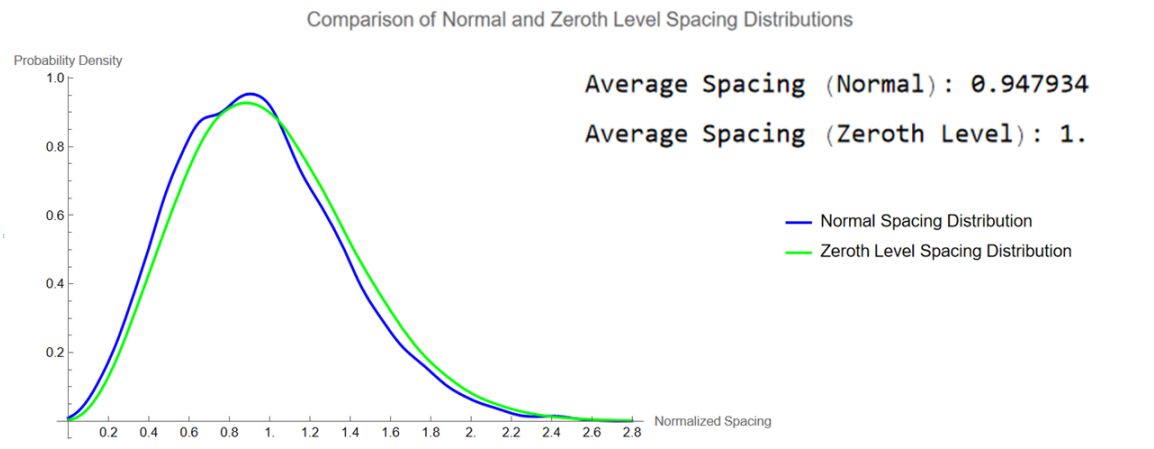

Figure 4.2: Comparison with fix point

In this particular graph the consecutive spacing is being taken from the circular orthogonal ensemble and the distribution of them is plotted on the graph, for comparison the green line denotes the distribution of zeroth local level spacing means the spacing for 1000 matrices in which the reference point would lie, while the blue line represents the normal spacing of circular orthogonal ensemble, as we can see in the figure the average spacing in which the reference point will lie is grater than that of the normal spacing.

Chapter 5

Distribution of Spacing Ratios With and Without Fix Point

In this particular section we will see the distribution of spacing ratios in circular orthogonal ensemble and compare their average values from that of predicted by the Bogomonly formulation, these values of ratio dataset is generated in Mathematica language. Reference point in this particular study is taken to be radian 4, with which all the analysis of graphs is done. Here we will study for the circular orthogonal ensembles, this because circular ensembles are more mathematically elegant, symmetry is there so no preferred basis is required. Also allows computation of correlation functions and spacing statistics explicitly.

Consider the circular Orthogonal ensemble that is what our point of study in this as we know the matrices are unitary matrices now their eigenvalues can be written in the form of eigen angles and also those angles can be distributed or spread on the circumference of unit circle as such the case we have considered in which we have taken the eigen angle spacing for matrix of 1000 by 1000 dimension plot their distribution in comparison with the Bogomonly's analytical formula and the ratio with reference point.

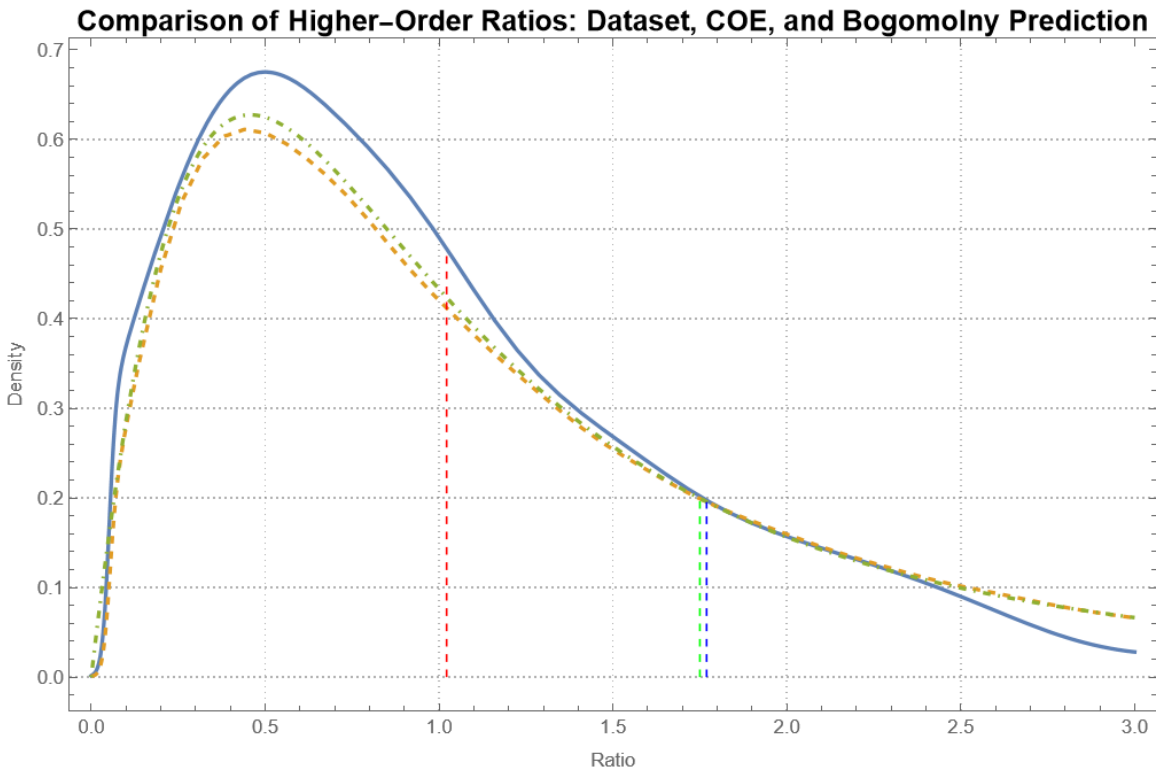

Figure 5.1: Comparison(1) between the distribution for ratios of spacing with fix point,without fix point and bogomonly curve.

In figure 5.1 the blue line represent the distribution of Circular orthogonal eigen angles spacing ratio of the spacing which holds the the reference point as here we have taken the angle in radian as 4, to understand this lets suppose the eigen angles are l_1, l_2, l_3, ... and are arranged in the ascending order now if we take one angle or not necessary it has to be eigen angle but yes in the range so in our case we take as 4 now the spacing which hold this is called as zeroth local level spacing and the consecutive spacing as local level spacing now the in this it is the distribution of $\frac{l_2-l_1}{l_1-l_0}$ that is shown by the blue line while yellow and green doted lines represents the ratio of consecutive level spacing and the Bogomonly prediction about it. We can conclude from here that the average value of local level spacing ratios and the Bogomonly predicted average is very close 1.75, also COE ratios are matching with this value it is approxx 1.77 as predicted by Bogomonly [2], while the spacing ratio with the reference point has unpredictable value that is approximately close to 1.

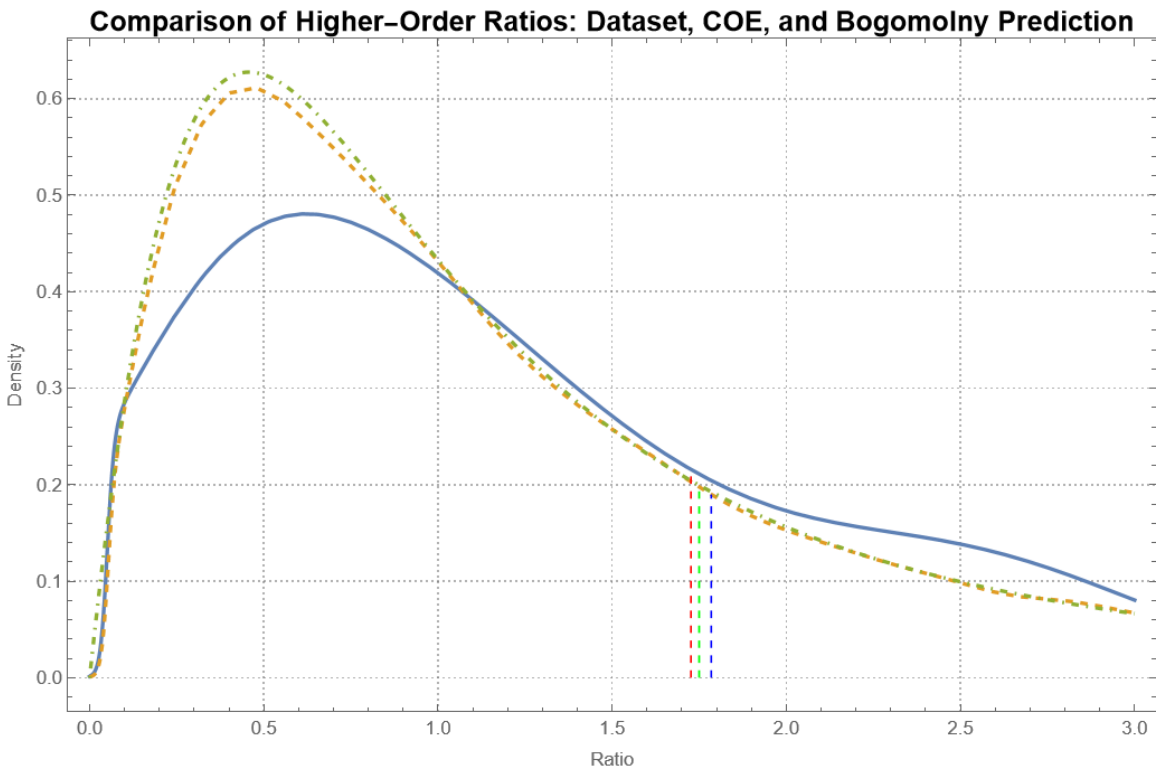

Figure 5.2: Comparison(2) between the distribution for ratios of spacing with fix point,without fix point and bogomonly curve.

Figure 5.2 gives a comparison for $< \frac{l_3 - l_2}{l_2 - l_1} >$ as can be called as $< r_1^{\text{loc}} >$ and the average value for $< r_1^{\text{loc}} >$ is 1.73 as it has shifted towards higher values.

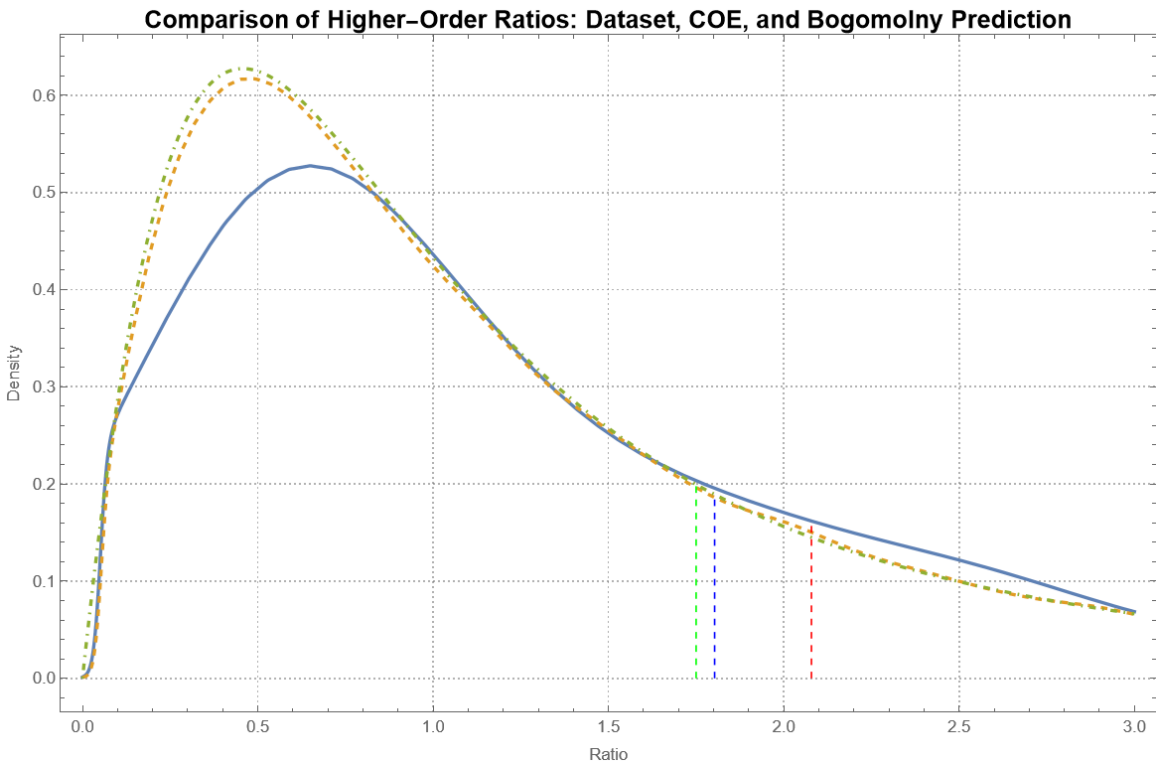

Figure 5.3: Comparison(3) between the distribution for ratios of spacing with fix point,without fix point and bogomonly curve.

Fig 5.3 gives a comparison for $< \frac{l_4-l_3}{l_3-l_2} >$ as can be called as $< r_2^{\text{loc}} >$ and the average value for $< r_2^{\text{loc}} >$ is approx 2.1 as it has shifted towards higher values.

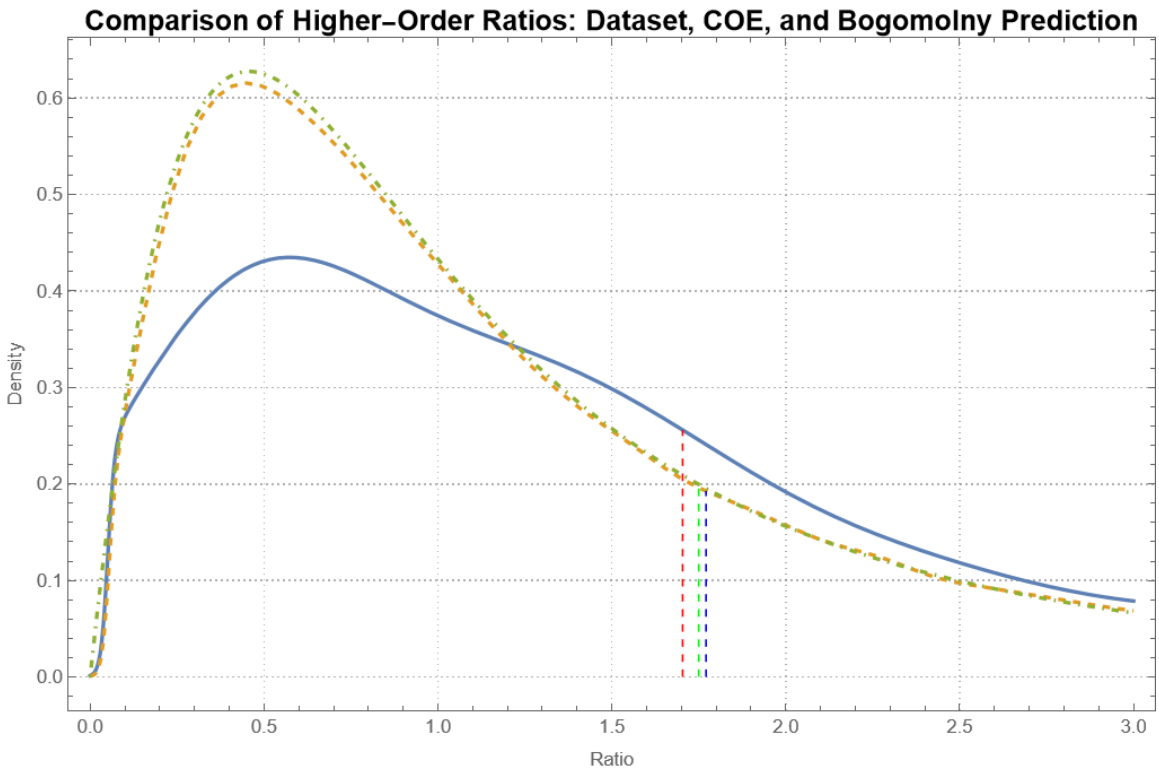

Figure 5.4: Comparison (4) between the distribution for ratios of spacing with fix point,without fix point and Bogomonly curve.

Fig 5.4 gives a comparison for $< \frac{l_5 - l_4}{l_4 - l_3} >$ as can be called as $< r_3^{\text{loc}} >$ and the average value for $< r_3^{\text{loc}} >$ is approx 1.71 as it has shifted towards higher values.

Chapter 6

Comparison with Analytical Formula by Bogomonly

The Study of ratios as is very important in the studies of spectral data of complex systems so the behavior should be analytically described such a formula for the gaussian ensembles was given by the Bogomonly in his formula he predicted the behavior of the ratios of local consecutive spacing that fits with his analytical formula stated as :

$$P(r) = \frac{1}{Z} \cdot \frac{(r + r^2)^{\beta}}{(1 + r + r^2)^{1+\frac{3}{2}\beta}}. \tag{6.1}$$

where $\frac{1}{Z}$ is the normalization constant and r is the ratios of consecutive spacing while β is the Dyson index which represent the order of level repulsion for circular orthogonal ensemble analogical to gaussian orthogonal ensemble value of β is 1 [2].

As from the plots we have done shown in **Chapter 5**, we can see that the ratio average with the reference point is different from the one without the reference point, so that can not be described by this analytical formula given by the Bogomolny.

By all that we have concluded, our results are in the table as

	$< r_0^{\text{loc}} >$	$< r_1^{\text{loc}} >$	$< r_2^{\text{loc}} >$	$< r_3^{\text{loc}} >$
for β=1	1.01	1.73	2.1	1.71

Table 6.1: Results

6.1 Conclusion and future directions

The study of the ratios of spacing with and without reference point is being studied in this work and it is found that with reference point the average values of ratios do not coincide with the analytical distribution formula given by Bogomonly, The values thus obtained need to be fit analytically while these values are significant in study of spectral behavior of systems which holds rotational invariance and do lie in Circular orthogonal ensemble. Table 6.1 illustrates a general upward trend in the local ratio values in $< r_n^{\text{loc}} >$ n increases. Interestingly, this pattern is interrupted at $n = 3$, where the value unexpectedly declines to around 1.71. Such irregular behavior highlights the complexity of the trend and emphasizes the importance of analyzing the spectral distribution of high-dimensional matrices more thoroughly to potentially establish a reliable analytical expression for the ratio distribution.

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